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Singularities in Optimization of Deterministic Dynamic Systems

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I. Introduction

FROM an historical standpoint scientific optimization studies are a relatively recent development. One of the best known examples is the determination of the optimal transfer orbit between two planets by Hohmann in 1925.[1] Until the beginning of the space era such studies remained rare and primarily theoretical.[2-6] Fortunately, the exploration of space has provided an excellent field for the development of optimization theories because space missions are extremely expensive and because simple and accurate mathematical models of space navigation are available.

At present, the field of optimization has literally exploded in all directions. It has led to new theoretical approaches as well as to the development of many numerical methods more or less based on these theories. The first numerical methods were very general but they suffered from a lack of convergence and were often not able to deal with the constraints of optimization problems (singularities, forbidden zones, etc.). They have been improved considerably over the years and are now employed in virtually all branches of science and engineering.

On the other hand, the optimization theories have led to the discovery of many types of singularities and even to some very surprising phenomena, such as those arising in the Fuller problem.[7] The existence of switching points, intermediate thrust arcs, singular arcs of various kinds, chattering arcs, etc., finally lead to very complex solutions. All of these singularities occur in systems which may be classified as "dynamic systems" or "dynamical systems," i.e., maneuverable and deterministic systems governed by ordinary differential equations. Natural extensions of these systems and their optimization would be differential games, stochastic optimization, and optimization of distributed parameter systems. However, we will not consider these last three subjects since they lead to an extreme variety of singularities (for differential games these include universal surfaces, dispersal surfaces, transition surfaces, focal lines, equivocal lines, barriers, bundles of trajectories, etc.). This survey will be restricted to the singularities occurring in the optimization of the above-defined dynamic systems.

II. Problem Statement

A. Dynamic Systems under Consideration

These systems can be described in a variety of ways as follows:

1) There is an independent variable t generally called time. Thus, a property which is true for almost all t is "almost always" true but not "almost everywhere" true.

2) The other parameters of interest, $x_1,x_2,\ldots,x_n$, are the components of the "state vector"

$$\vec{x}=(x_1,x_2,\ldots,x_n) \tag{1}$$

3) The evolution of $\vec{x}$ is controlled by a "control vector" $\vec{u}$ and a "control function" (or even a "control vector-function") f

$$\dot{\vec{x}}=\mathrm{d}\vec{x}/\mathrm{d}t=f(\vec{x},\vec{u},t) \tag{2}$$

4) The control vector $\vec{u}$ can be chosen anywhere in a given "control domain" Δ

$$\vec{u}\in\Delta(\vec{x},t) \tag{3}$$

Usually Δ is independent of $\vec{x}$ and t.

5) It is possible to disregard the control vector $\vec{u}$ and to consider the "maneuverability domain" D, such that

$$\dot{\vec{x}}\in D(\vec{x},t) \tag{4}$$

which obviously implies that

$$D(\vec{x},t)=f\{\vec{x},\Delta(\vec{x},t),t\} \tag{5}$$

6) The definition of the admissibility of a trajectory $\vec{x}(t)$ can now be considered, i.e., its compatibility with Eqs. (2) and (3) [or with Eq. (4)]. Discontinuities of $\vec{x}(t)$ in relation to the "control impulses" are possible. If so, "admissibility of the discontinuous type" is also possible.[8] However, the local compactness of the set of admissible trajectories requires

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more conditions for $f(\vec{x},\vec{u},t)$, $\Delta(\vec{x},t)$, and $D(\vec{x},t)$ than are required for the "admissibility of the continuous type"[8] in which the continuity of $\vec{x}(t)$ is needed.

We will not consider the admissibility of the discontinuous type since it can be brought back to the continuous type when taking another independent variable into account.

Use of "piecewise-continuous controls" $\vec{u}(t)$ assumes that $\vec{x}(t)$ is admissible if:

$$\dot{\vec{x}}(t)=f\{\vec{x}(t),\vec{u}(t),t\} \qquad \text{(almost always)}$$

$$\vec{u}(t)\in\Delta\{\vec{x}(t),t\} \qquad \text{(almost always)} \tag{6}$$

Note that the continuity and the almost always differentiability of $\vec{x}(t)$ are not sufficient to give a valid definition since there exist continuous and nonconstant functions which have almost always a zero derivative. To be more specific: $\vec{x}(t)$ should be "absolutely continuous." It is an integral of its derivative (with the very general measure and integral of Lebesgue):

$$\vec{x}(t_2)-\vec{x}(t_1)=\int_{t_1}^{t_2}\dot{\vec{x}}(t)\,\mathrm{d}t \tag{7}$$

The restriction to "piecewise-continuous controls" $\vec{u}(t)$ is unfortunate; it leads to difficulties in problems as simple as that of Fuller (see example 16 in Sec. VI. G). A wider and better choice is that of measurable controls $\vec{u}(t)$.

If we use the maneuverability domain D defined in Eq. (4), the admissibility is very simple: $\vec{x}(t)$ is an admissible trajectory if and only if

$$\vec{x}(t) \quad \text{is absolutely continuous}$$

$$\dot{\vec{x}}(t)\in D\{\vec{x}(t),t\} \quad \text{(almost always)} \tag{8}$$

The absolute continuity of $\vec{x}(t)$ is determined as follows: Assume an arbitrary compact interval $[t_1,t_2]$ in which $\vec{x}(t)$ is defined. Also assume an arbitrary sequence of disjoint subintervals (t_a,t_b), (t_c,t_d),..., and the corresponding $\vec{x}$ intervals $(\vec{x}_a,\vec{x}_b)$, $(\vec{x}_c,\vec{x}_d)$,.... Then $\vec{x}(t)$ is absolutely continuous in $[t_1,t_2]$ if, with some usual definition of the norm, the sum $\|\vec{x}_a-\vec{x}_b\|+\|\vec{x}_c-\vec{x}_d\|+\ldots$ goes to zero along with the sum $|t_a-t_b|+|t_c-t_d|+\ldots$. Thus, $\vec{x}(t)$ is absolutely continuous if it has this property in any compact interval in which it is defined. The absolute continuity implies the almost always differentiability of $\vec{x}(t)$, and equivalent to the condition of Eq. (7).

The admissibility defined by Eq. (8) is equivalent to that defined by Eqs. (6) and (7) for measurable $\vec{u}(t)$, provided that Eqs. (2) and (3) are Borel-measurable relations, as is indeed the case in all usual problems.

B. Optimization Problems of Dynamic Systems

Just like the corresponding dynamic systems, these optimization problems present themselves in a large variety of ways. We will describe them in as general a way as possible.

1) There is a performance index I:

$$I=I_1(\vec{x}_0,t_0,\vec{x}_f,t_f)+\int_{t_0}^{t_f} I_2\{\vec{x}(t),\vec{u}(t),t\}\,\mathrm{d}t \tag{9}$$

2) There is an "authorized domain" $\mathfrak{D}$ in the R^{2n+2} space of end states and times:

$$(\vec{x}_0,t_0,\vec{x}_f,t_f)\in\mathfrak{D} \tag{10}$$

For instance, $\vec{x}_0,t_0$, and t_f are given and $\vec{x}_f$ is free.

3) The optimization problem consists of finding the admissible trajectory $\vec{x}(t)$ leading from $\vec{x}_0$ at t_0 to $\vec{x}_f$ at t_f, satisfying Eq. (10) and giving to I its maximum (or minimum) possible value. In the case of a minimization, I is usually called a cost function.

4) Sometimes, there are some additional constraints, e.g., the occurrence of a forbidden zone $\mathfrak{F}$ in the $\vec{x},t$ space. For all t within $[t_0,t_f]$:

$$\{\vec{x}(t),t\}\notin\mathfrak{F} \tag{11}$$

Note that the forbidden zones for the control $\vec{u}$ were already implied by Eq. (3).

Classically, an optimization problem is called: 1) the "Lagrange problem" if $I_1\equiv 0$ in Eq. (9); 2) the "Mayer problem" if $I_2\equiv 0$ in Eq. (9); 3) the "Bolza problem" in the general case, but as will be shown in Sec. IV the Mayer problem has in fact the same degree of generality; or 4) the "linear quadratic problem" if the following conditions are satisfied:

The control domain $\Delta(\vec{x},t)$ is an R^m space (12a)

$$f(\vec{x},\vec{u},t)=F(t)\vec{x}+G(t)\vec{u} \tag{12b}$$

The authorized domain $\mathfrak{D}$ gives $\vec{x}_0,t_0,t_f$, and some linear relations on $\vec{x}_f$ (12c)

I is to be minimized: (12d)

$$I_1=\vec{x}_f^T S\vec{x}_f$$

$$I_2=\vec{x}^T Q(t)\vec{x}+2\vec{x}^T H(t)\vec{u}+\vec{u}^T R(t)\vec{u}$$

The matrices $F(t)$, $G(t)$, S, $Q(t)$, $H(t)$, and $R(t)$ are given with $R(t)\geq 0$ (12e)

The forbidden zone $\mathfrak{F}$ is empty (12f)

Linear quadratic problems are very useful in the study of the immediate vacinity in a given trajectory $\vec{x}(t)$.

III. The Singularities

We will classify the singularities with respect to their nature: 1) the topological singularities; 2) a singularity related to the optimization procedure that appears in problems of the Lagrange or Bolza form but disappears when the problem is brought back to the Mayer form; and 3) the singularities related to the optimization itself (singular arcs and their "generalized Legendre-Clebsch condition," intermediate thrust arcs, and chattering arcs of the second kind).

IV. The Topological Singularities

When considering a given problem, the usual questions are: Has the problem a solution? Is that solution unique? Are there some simple conditions which must necessarily be satisfied by the solution and is there a general way to find the latter? What are the singular cases and the singularities? How are they dealt with? Finally, which analytical or numerical methods are able to find the solution with any given accuracy? What is their domain of application? What are their difficulties of convergence?

The question of uniqueness is often considered of secondary importance, but the question of existence is central. For linear quadratic problems, it is strongly related to the ideas of robustness, controllability, and extendability.[9-16] Hence, the topological analysis is as follows:

1) Is the problem feasible? That is, does there exist an admissible trajectory $\vec{x}(t)$, even if nonoptimal, satisfying the end conditions of Eq. (10) and other constraints such as Eq. (11)? This question is easy in general and is related to the controllability of linear quadratic systems.

2) Consider the candidate solutions and the corresponding values of the performance index I. We are looking for the

optimal value of I, so the second question is: Does an admissible trajectory correspond to the optimal value of I? We are inclined to take as the optimal solution the limit of a sequence of the best candidate solutions, and the question is that of the existence and the admissibility of the limit.

This question arises very often in mathematics and is related to the convergence of a given sequence and thus to the local compactness of the set of elements of interest, i.e., the set of admissible trajectories and their performance index. Very simple results can be obtained if the problem has a Mayer form.

As discussed in Sec. II. B the Mayer form corresponds to an I_2 function identically zero in Eq. (9). If this is not the case, it is straightforward to obtain that form by adding a supplementary parameter x_{n+1} to the state vector $\vec{x}$, a parameter x_{n+1} such that

$$x_{n+1}(t) = \int_{t_0}^{t} I_2\{\vec{x}(\theta),\vec{u}(\theta),\theta\}\,d\theta \tag{13}$$

Thus the initial value of x_{n+1} is zero and its control function is

$$dx_{n+1}/dt = I_2\{\vec{x}(t),\vec{u}(t),t\} \tag{14}$$

The performance index I now has the Mayer form

$$I = I_1(\vec{x}_0,t_0,\vec{x}_f,t_f) + x_{n+1,f} \tag{15}$$

With this Mayer form and with one of the usual definitions of the distance between two trajectories, the set of admissible trajectories and their performance index is locally compact if the conditions of partial canonicity in the meaning of Contensou[8] are satisfied, i.e.:

The forbidden zone $\mathfrak{F}$ of Eq. (11) is open (e.g., it is empty) and the authorized domain $\mathfrak{D}$ of Eq. (10) is closed. (16a)

The performance index I, now equal to $I(\vec{x}_0,t_0,\vec{x}_f,t_f)$, is a continuous function of $\vec{x}_0,t_0,\vec{x}_f$, and t_f. (16b)

For almost all t the maneuverability domain $D(\vec{x},t)$ defined in Eq. (5) is closed, convex, and upper semicontinuous with respect to $\vec{x}$ [i.e, $\vec{x}_n \to \vec{x}; \dot{\vec{x}}_n \in D(\vec{x}_n,t); \dot{\vec{x}}_n, \to \vec{v}$ implies $\dot{\vec{x}} \in D(\vec{x},t)$]. (16c)

For a bounded set B of the $\vec{x},t$ space, let $W_B(t)$ be the largest velocity $\|\dot{\vec{x}}(t)\|$ at instant t for admissible trajectories of set B. (16d)

The last condition is then:

For any bounded set B, the function $W_B(t)$ is integrable.

With reference to Eqs. (16), the following should be noted:

Note 1. These conditions of local compactness correspond to the admissibility conditions of measurable controls $\vec{u}(t)$. If only piecewise-continuous controls $\vec{u}(t)$ are allowed, it becomes very difficult to be sure of the existence of an optimal solution (see example 16 in Sec. VI.G and Refs. 7 and 17).

Note 2. Since we are looking for the optimal value of the performance index I, its continuity rule can be relaxed. For instance, if I is to be maximized, it is sufficient that $I(\vec{x}_0,t_0,\vec{x}_f,t_f)$ be an upper semicontinuous function.

Note 3. As usual, the local compactness of the set of interest assures the existence of a limit (i.e., the optimal solution) only if the sequence is "local," i.e. only if the best nonoptimal solutions belong to some bounded set of the $\vec{x},t$ space.

Thus, the remote parts of the authorized domain $\mathfrak{D}$ of Eq. (10) must be either uninteresting (bad performance index) or unattainable. Furthermore, the trajectories "going as far as desired and coming back in a finite time" must be either impossible or uninteresting. Consider the following counterexample.

Example 1. Optimal trajectories "going as far as desired and coming back in a finite time."

Control function and control domain: $\dot{x} = ux^2; \quad -1 \le u \le +1$ (17a)

End conditions: $x_0 = x_f = 1; \quad t_0 = 0; \quad t_f = 3$ (17b)

Question: minimize $I = \int_0^3 \frac{dt}{x^2}$ (17c)

The greatest lower bound of I is 2/3 and the sequence of best candidate solutions can be the following in terms of the positive parameter ϵ:

$0 \le t \le 1-\epsilon$:	$u=1$	$x = 1/(1-t)$
$1-\epsilon < t < 2+\epsilon$:	$u=0$	$x = 1/\epsilon$
$2+\epsilon \le t \le 3$:	$u=-1$	$x = 1/(t-2)$

$$I = (2+3\epsilon^2+4\epsilon^3)/3 \tag{18}$$

When $\epsilon \to 0_+$ the performance index I goes to its greatest lower bound 2/3 but the solutions go to infinity.

Note 4. The linear quadratic problems satisfy the conditions of Eqs. (16a) and (16b), but neither the condition of Eq. (16d) nor, when in the Mayer form, the convexity condition of Eq. (16c).

The convexization of the maneuverability domain does not alter the result: it cannot lead to better I (which is related to the property called "directional convexity"), but the difficulties related to the nonverification of the condition of Eq. (16d) have led to various tests of "robustness," "extendability," "non-negativity," or "finiteness."[9-16]

The following examples may explain this more clearly.

Example 2. Optimal trajectories "going as far as desired and coming back in a finite time" in a linear quadratic problem.

Control function and control domain: $\dot{x} = u; \quad -\infty < u < +\infty$ (19a)

End conditions: $x_0 = t_0 = 0; \quad x_f = 1; \quad t_f$ given and positive (19b)

Question: minimize $I = \int_{t_0}^{t_f} (u^2 - x^2)\,dt$ (19c)

If $t_f < \pi$ the optimal solution is bounded:

$$x = \sin t/\sin t_f \qquad I = \cot t_f \tag{20}$$

hence when $t_f \to \pi$ the optimal I goes to minus infinity.

When $t_f \ge \pi$, the performance index I may have any value and its greatest lower bound is minus infinity and the optimization leads to unbounded trajectories. This property is analyzed by the tests of "non-negativity" and of "finiteness"[9-16].

Example 3. Discontinuities in linear quadratic problems.

Control function and control domain: $\dot{x} = u; \quad -\infty < u < +\infty$ (21a)

End conditions: $x_0 = t_0 = 0;\ x_f = t_f = 1$ (21b)

Question: minimize $I = \int_{t_0}^{t_f} tu^2 dt$ (21c)

The greatest lower bound of I is zero and it can be shown with the following sequence of continuous solutions in terms of the parameter K:

$$K > 0 \quad (22a)$$

$$0 \le t < \exp\{-K\} \Rightarrow u = x = 0 \quad (22b)$$

$$\exp\{-K\} \le t \le 1 \Rightarrow u = 1/Kt \qquad x = 1 + \ln t/K \quad (22c)$$

$$I = 1/K \quad (22d)$$

When K goes to infinity, the performance index I goes to zero; however, these continuous solutions have a discontinuous limit with a "control impulse" leading x from zero to one at $t = 0$, and, also, it is obvious that no continuous solution can give $I = 0$.

However, it is very difficult to consider discontinuous solutions as something more than merely a limit of a proper sequence of continuous solutions, for example:

Example 4. A problem very similar to example 3.

Control function and control domain: $\dot{x} = u;\ -\infty < u < +\infty$ (23a)

End conditions: $x_0 = t_0 = 0;\ x_f = t_f = 1$ (23b)

Question: minimize $I = \int_{t_0}^{t_f} u^2 \sqrt{t} dt$ (23c)

The best continuous solution is now: $u = 1/(2\sqrt{t})$ and $x = \sqrt{t}$, which gives $I = ½$. Hence, in spite of $u^2\sqrt{t} = 0$ at $t = 0$, it is impossible to consider the discontinuous solution of example 3 as realistic.

A general analysis of this difficult question leads to the "admissibility of the discontinuous type."[8] The admissible trajectories are then not necessarily absolutely continuous, but they have a bounded total variation (and thus at most a denumerable number of discontinuities with some obvious rules for these discontinuities). The local compactness of the set of these solutions requires the conditions of Eqs. (16a) and (16b) but Eq. (16d) is no longer necessary and Eq. (16c) becomes:

> The maneuverability domain $D(\vec{x},t)$ is closed convex and upper semicontinuous with *respect to* $\vec{x}$ *and* t. (24)

Condition (24) is more stringent than Eq. (16c) and generally implies the continuity of the matrices F, G, Q, H, and R of Eq. (12) with respect to t. The extension of the Pontryagin maximum principle to this type of admissibility is not easy.

Note 5. There is a simple case in which these difficulties of discontinuities are avoided.

We have seen in Eq. (12) that one of the conditions for linear quadratic problems is $R(t) \ge 0$. This means that the matrix $R(t)$ is symmetric and non-negative definite and that its eigenvalues are real and non-negative. In a given feasible linear quadratic problem, with $r(t)$ denoting its smallest $R(t)$ eigenvalue, assume that 1) the matrices $F(t)$, $G(t)$, $Q(t)$, $H(t)$, and $R(t)$ are, as usual, measurable functions of t; 2) the function $1/r(t)$ is an *integrable* function of t; and 3) the topological singularities of Note 3 are avoided. Then the set of solutions is locally compact and the optimal solution of the problem of interest is unique and absolutely continuous, and can thus be obtained by the Pontryagin theory of optimization (see Sec. VI.A or Ref. 17).

Note 6. This review of topological singularities can be concluded with an analysis of the convexity condition of Eq. (16c). The conditions of local compactness in Eq. (16) are quite obvious except perhaps for the convexity condition which leads to two questions: What happens to systems with nonconvex maneuverability domains? and Is it justified to "convexize" the maneuverability domains (i.e., to "relax" control[18-22])?

Dynamic systems with nonconvex maneuverability domains lead to "chattering solutions of the first kind."[20] The optimal control $\vec{u}$ and the corresponding velocity $\dot{\vec{x}}$ "chatter" at arbitrarily rapid rates between two or more values such as $\dot{\vec{x}}_A$ and $\dot{\vec{x}}_B$ (Fig. 1) and it gives an intermediate velocity such as $\dot{\vec{x}}_C$, as in the following example.

Example 5. Nonconvex maneuverability domain.

Control function and control domain: $\dot{x} = u = \pm 1$ (25a)

Question: minimize $I = \int_0^1 x^2 dt$ (25b)

The greatest lower bound of I is zero but it can be obtained only by a relaxation of the control that allows $\dot{x} = 0$. The "convexization" and the closure of the maneuverability domain are not always justified – they can lead to solutions which are not the limit of earlier solutions as in the following examples.

Example 6. Unjustified closure.

Maneuverability domain: $2\sqrt{|x|} < \dot{x} \le 1 + 2\sqrt{|x|}$ (26a)

End conditions: $x_0 = t_0 = 0;\ t_f = 1$ (26b)

Question: minimize x_f (26c)

Before closure, the greatest lower bound of x_f is 1; after closure, the solution $x(t) = 0$ for $0 \le t \le 1$ becomes admissible.

Example 7. Unjustified convexization and relaxation.[19,20]

Control function and control domain:

$$\dot{x}_1 = u = \pm 1; \quad \dot{x}_2 = x_1^2 + 2\sqrt{|x_2|} \quad (27a)$$

End conditions: $x_{1,0} = x_{2,0} = t_0 = 0;\quad x_{1f} = 0;\quad t_f = 1$ (27b)

Question: minimize x_{2f} (27c)

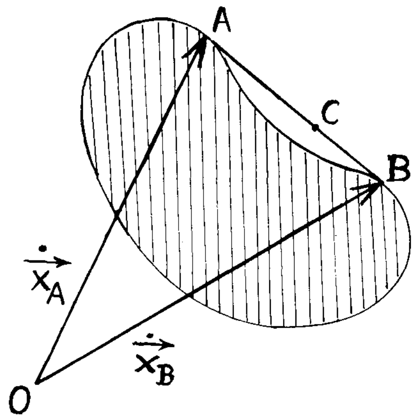

Fig. 1 Shaded nonconvex maneuverability domain and its "convexization" along AB either by "chattering" or by "relaxation."

This example is very similar to the previous one: before convexization the greatest lower bound of x_{2f} is 1; after convexization, $\dot{x}_1 = 0$ becomes possible as well as the solution $\vec{x}=0$ for $0 \le t \le 1$.

We will present in Sec. VI.A a general rule, the "generalized canonicity of Pontryagin,"[8] under which the relaxation of the control and the closure and convexization of the maneuverability domains become justified and lead to trajectories which are the limit of suitable sequences of initially admissible trajectories.

V. An Artificial Singularity Related to the Optimization Procedure

The assumption that singularity appears for problems put in the Lagrange or Bolza forms but disappears in the Mayer form is made here. Consider the following example.

Example 8. An artificial singularity.

Control function: $\dot{x}_1 = \cos u_1; \quad \dot{x}_2 = \sin u_1; \quad \dot{x}_3 = u_2$ (28a)

Control domain: $-\pi \le u_1 \le +\pi; \quad -1 \le u_2 \le +1$ (28b)

End conditions: $x_{1.0} = x_{2.0} = x_{3.0} = t_0 = 0;$

$x_{1f} = t_f = 1; \quad x_{2f}$ and x_{3f} are free (28c)

Question: maximize $I = \int_0^1 (x_2 + x_3)\,dt$ (28d)

Since $\dot{x}_1 = \cos u_1 \le 1$ and $t_f - t_0 = x_{1f} - x_{1.0}$, we almost always need $\cos u_1 = 1$ and thus $\sin u_1 = 0$; on the other hand, we obviously almost always need $u_2 = 1$ and thus $x_1 = x_3 = t$, $x_2 = 0$, and $I = 0.5$.

Let us now analyze the problem by the usual Hamiltonian procedure of the Bolza form:

$$H = p_1\dot{x}_1 + p_2\dot{x}_2 + p_3\dot{x}_3 + x_2 + x_3$$

$$= p_1\cos u_1 + p_2\sin u_1 + p_3 u_2 + x_2 + x_3 \qquad (29)$$

I is to be maximized, then $\vec{u} = \arg\cdot\max_{\vec{u}} H(\vec{p},\vec{x},\vec{u},t)$, i.e.:

$$\cos u_1 = p_1/\sqrt{p_1^2 + p_2^2} \qquad \sin u_1 = p_2/\sqrt{p_1^2 + p_2^2} \qquad u_2 = \mathrm{sign}\, p_3 \qquad (30)$$

and

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = 0 \qquad \dot{p}_2 = -\frac{\partial H}{\partial x_2} = -1 \qquad \dot{p}_3 = -\frac{\partial H}{\partial x_3} = -1 \qquad (31)$$

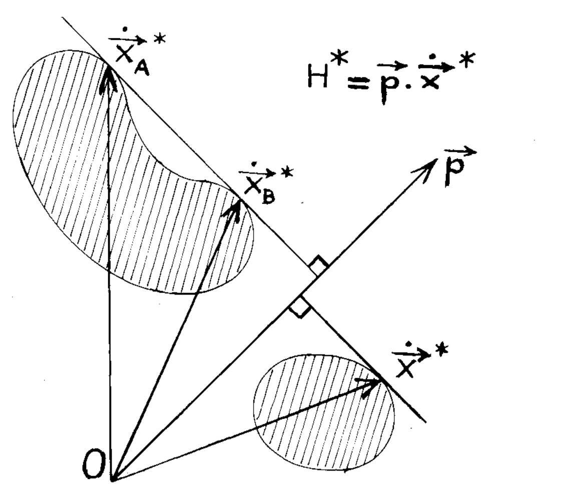

Fig. 2 Examples of shaded maneuverability domains (convex and nonconvex) and the corresponding generalized Hamiltonian.

where p_1 is constant, p_2 decreases, and hence $\sin u_1$ decreases. This behavior does not correspond to the optimal solution for which $\sin u_1$, $\dot{x}_2$, and x_2 are zero.

Finally, using the Mayer form

$$x_4(t) = \int_{t_0}^{t} \{x_2(\theta) + x_3(\theta)\}\,d\theta \qquad (32a)$$

$$x_{4.0} = 0 \qquad I = x_{4.f} \qquad (32b)$$

$$H = p_1\cos u_1 + p_2\sin u_1 + p_3 u_2 + p_4(x_2 + x_3) \qquad (32c)$$

The optimal solution corresponds to $p_1 = 1$, $p_2 = 0$, $p_3 = 1$, and $p_4 = 0$; thus, because $p_4 = 0$, the singularity is removed.

The usual procedure of the Bolza form can still be used if we allow for the possibilty of that zero factor in the $(x_2 + x_3)$ term, i.e., the I_2 term, of H.

VI. Singularities Related to the Optimization Itself

A. The Pontryagin Maximum Principle and Its Generalization

Because we want to analyze and survey the various singularities, it is necessary to know first the general rule established by many successive authors such as Weierstrass, McShane, Halkin, Roxin, Contensou, Breakwell, Lawden, Pontryagin[17,23-26] – a rule known as the Pontryagin maximum principle. In this section this principle will be presented in its most general and simplest form.

We assume that the performance index I is to be maximized (otherwise $-I$ is to be maximized) and we use the *Mayer form* of the problem in order to avoid the artificial singularity presented in Sec. V.

Note that with the Mayer form, the control $\vec{u}$ becomes only "a convenient way to describe the maneuverability." Any $\vec{v}$ may be substituted for $\vec{u}$ with a one-to-one correspondence and thus the problem can be separated into two questions: 1) that of the customer – What can be done and at what price? (i.e. determination of the optimal $\vec{x}(t)$ and I); and 2) that of the engineer – How can that optimal solution be achieved? (i.e. determination of the optimal control $\vec{u}(t)$). It is obvious that the second question is just a formality as soon as the optimal $\vec{x}(t)$ is known and so we will concentrate on the first question, disregarding the control $\vec{u}$ and using the maneuverability domain $D(\vec{x},t)$ of Eqs. (4) and (5).

Following Pontryagin, we introduce an adjoint vector $\vec{p}(t)$ and the corresponding Hamiltonian function H

$$H = \vec{p}\cdot\dot{\vec{x}} = \vec{p}\cdot\vec{f}(\vec{x},\vec{u},t) \qquad (33)$$

That Hamiltonian is called either "control Hamiltonian" or "generalized Hamiltonian" or only "Hamiltonian."

From H we define H^* by

$$H^* = \sup_{\dot{x}\in D(\vec{x},t)} \vec{p}\cdot\dot{\vec{x}} = \sup_{\vec{u}\in\Delta(\vec{x},t)} \vec{p}\cdot\vec{f}(\vec{x},\vec{u},t) \qquad (34)$$

H^* is thus a function of $\vec{p}$, $\vec{x}$, and t and is called "optimal Hamiltonian," "generalized Hamiltonian," or sometimes just "Hamiltonian" by various authors. All this is, of course, very confusing and thus we will use control Hamiltonian for $H(\vec{p},\vec{x},\vec{u},t)$ and optimal Hamiltonian for $H^*(\vec{p},\vec{x},t)$.

$H^*(\vec{p},\vec{x},t)$ is obviously independent of the closure and of the "convexization" of the maneuverability domain $D(\vec{x},t)$ (Fig. 2). Thus, closure and convexization need to be justified within the meaning of Note 6. This is the case if, for example, the three following conditions are satisfied (generalized canonicity of Pontryagin[8]):

The relation $\dot{\vec{x}}\in D(\vec{x},t)$ or else the two relations $\vec{u}\in\Delta(\vec{x},t)$ and $\dot{\vec{x}} = \vec{f}(\vec{x},\vec{u},t)$ are Borel measurable. (35a)

The forbidden zone $\mathfrak{F}$ of Eq. (11) is closed. (35b)

For any bounded set B of the $\vec{x},t$ space, there exists an integrable function $h(t)$ such that for almost all t: (35c)

$$\{(\vec{x},t)\in B-\mathfrak{F}\}\Rightarrow\Delta(\vec{x},t)\text{ is not empty}$$

$$\{(\vec{x},t)\in B-\mathfrak{F};\ \vec{u}\in\Delta(\vec{x},t)\}\Rightarrow\|\vec{f}(\vec{x},\vec{u},t)\|\leq h(t)$$

$$\{(\vec{x}_1,t)\in B-\mathfrak{F};\ \vec{u}_1\in\Delta(\vec{x}_1,t);\ (\vec{x}_2,t)\in B-\mathfrak{F}\}$$

$$\Rightarrow\exists\vec{u}_2\quad\in\Delta(\vec{x}_2,t)$$

$$\text{and such that }\|\vec{f}(\vec{x}_1,\vec{u}_1,t)-\vec{f}(\vec{x}_2,\vec{u}_2,t)\|$$

$$\leq\|\vec{x}_1-\vec{x}_2\|\cdot h(t)$$

In terms of the maneuverability, Eq. (35c) becomes for almost all t:

$$\{(\vec{x}_1,t)\text{ and }(\vec{x}_2,t)\in B-\mathfrak{F}\}\Rightarrow|H^*(\vec{p}_1,\vec{x}_1,t)-H^*(\vec{p}_2,\vec{x}_2,t)$$
$$\leq h(t)\{\|\vec{p}_1-\vec{p}_2\|+\|\vec{p}_1\|\cdot\|\vec{x}_1-\vec{x}_2\|\}\tag{36}$$

Thus $H^*(\vec{p},\vec{x},t)$ is almost always a local Lipschitz function with respect to $\vec{p}$ and $\vec{x}$, but it is measurable only with respect to t.

Note that, after convexization and closure of the maneuverability domains, Eqs. (35c) implies Eqs. (16c) and (16d). Furthermore, if Eqs. (16a) and (16b) are satisfied, the problem is then called "bicanonical"[27] and has many interesting properties that will be discussed in following sections.

Equations (16a) and (35b) imply that the forbidden zone $\mathfrak{F}$ is empty, but the problem remains bicanonical if $\mathfrak{F}$ covers the two half-spaces defined by $t<T_1$ and $t>T_2$. Now, the following definitions are needed:

1) Connectable couple $(\vec{x}_1,t_1)$, $(\vec{x}_2,t_2)$ – A couple such that an admissible trajectory $\vec{x}(t)$ leads from $(\vec{x}_1,t_1)$ to $(\vec{x}_2,t_2)$.
2) Limit couple – A connectable couple, but with nonconnectable couples in any of its vicinities.
3) Limit trajectory – Its end couple $(\vec{x}_0,t_0)$, $(\vec{x}_f,t_f)$ is a limit couple.
4) Extremal trajectory – All its couples $(\vec{x}(t_1),t_1)$, $(\vec{x}(t_2),t_2)$ are limit couples.
5) Locally limited or locally extremal trajectories – These become limit or extremal if we consider only one of their sufficiently small neighborhoods.

For bicanonical systems, all limit trajectories are also extremal and all locally limit trajectories are also locally extremal. The Pontryagin maximum principle can be extended to all bicanonical systems and it can be expressed in the following way, independently of the control $\vec{u}$. For each locally extremal trajectory $\vec{x}(t)$ defined on $[t_0,t_f]$, there corresponds at least one "adjoint function" $\vec{p}(t)$ satisfying the following properties:

$$\vec{p}(t)\text{ is absolutely continuous and nonzero on }[t_0,t_f]\tag{37}$$

At almost all points $\{\vec{p}(t),\vec{x}(t),t\}$ where $H^*(\vec{p},\vec{x},t)$ is continuously differentiable with respect to $\vec{p}$ and $\vec{x}$, one has

$$\dot{\vec{x}}=\partial H^*/\partial\vec{p}\qquad\dot{\vec{p}}=-\partial H^*/\partial\vec{x}\tag{38}$$

At almost all other points (i.e., usually on some submanifolds)

$$(\dot{\vec{x}},-\dot{\vec{p}})\in D_H(\vec{p}(t),\vec{x}(t),t)\tag{39}$$

This domain D_H gives a generalization of Eq. (38). It is the closed convex hull of local gradients (with $2n$ components) $\partial H^*/\partial(\vec{p},\vec{x})$ taken at the points $\{\vec{p}(t)+\delta\vec{p},\vec{x}(t)+\delta\vec{x},t\}$ where $\delta\vec{p}$ and $\delta\vec{x}$ are infinitesimal variations and H^* is differentiable with respect to $\vec{p}$ and $\vec{x}$ (i.e., D_H is the limit for $\epsilon\to0$ of the domains $D_{H\epsilon}$ obtained when $\|\delta\vec{p}\|<\epsilon$ and $\|\delta\vec{x}\|<\epsilon$).

Note that Eqs. (34), (38), and (39) almost always imply the usual "condition of the maximum," that is

$$\vec{p}\cdot\dot{\vec{x}}=\sup_{\dot{\vec{x}}\in D(\vec{x},t)}\vec{p}\cdot\dot{\vec{x}}=H^*\tag{40}$$

We will call D_V the projection of D_H on the space of velocities $\dot{\vec{x}}$; it is also the closed and convex set of velocities satisfying the condition of the maximum [Eq. (40)] and belonging to the closed convex hull of the maneuverability domain $D(\vec{x},t)$:

$$\{\dot{\vec{x}}\in D_V(\vec{p},\vec{x},t)\}$$
$$\Leftrightarrow\{\vec{p}\dot{\vec{x}}=H^*(\vec{p},\vec{x},t),\text{ and }\dot{\vec{x}}\in\text{ closed convex hull of }D(\vec{x},t)\}\tag{41}$$

This generalization of Pontryagin's maximum principle is very useful because it remains valid even for singular arcs. A few examples are discussed below.

Note 7. If $\vec{p}(t)$ satisfies the conditions of Eqs. (37-39), the function $\lambda\vec{p}(t)$ also satisfies these conditions (λ being an arbitrary *positive* constant) and the functions $\vec{p}(t)$ and $\lambda\vec{p}(t)$ are not considered as independent.

Note 8. The Pontryagin maximum principle is related to the first-order analysis of the vicinity of the solution $\vec{x}(t)$ of interest, and so is its generalization. It implies that if only one independent adjoint function $\vec{p}(t)$ corresponds to the extremal trajectory $\vec{x}(t)$ of interest, the neighboring end couples, $(\vec{x}_0+\delta\vec{x}_0,t_0)$ and $(\vec{x}_f+\delta\vec{x}_f,t_f)$, are connectable by neighboring trajectories as soon as

$$\vec{p}_0\cdot\delta\vec{x}_0\geq\vec{p}_f\cdot\delta\vec{x}_f+o(\delta\vec{x}_0,\delta\vec{x}_f)\tag{42}$$

but the other neighboring couples can also sometimes be connectable since the Pontryagin conditions are the only necessary conditions of local extremality.

Note 9. If $H^*(\vec{p},\vec{x},t)$ is continuous at $(\vec{p}_0,\vec{x}_0,t_0)$ and at $(\vec{p}_f,\vec{x}_f,t_f)$, then Eq. (42) can be extended to the usual expression

$$\vec{p}_0\cdot\delta\vec{x}_0-H_0^*\delta t_0\geq\vec{p}_f\cdot\delta\vec{x}_f-H_f^*\delta t_f+o(\delta\vec{x}_0,\delta t_0,\delta\vec{x}_f,\delta t_f)\tag{43}$$

This leads to the usual transversality conditions.

Note 10. If more than one independent adjoint function $\vec{p}(t)$ corresponds to the extremal trajectory of interest, the analysis is more complex but remains classical and can sometimes be improved in various ways.

Note 11. In linear quadratic problems the above conditions of bicanonicity are never satisfied because the possible velocities $\dot{\vec{x}}$ are always unbounded. These problems have received considerable attention[9-16] and various tests of applicability of the Pontryagin maximum principle have been defined (e.g., when the conditions of Note 5 are satisfied).

B. Singular Arcs and Singular Controls

We have discussed the topological singularities in Sec. IV and an artificial singularity in Sec. V. Let us now consider the most interesting aspect of this survey – the singularities related to the optimization itself, especially the singular arcs and singular controls.[10-16,28-70]

The prototype of singular arcs is the Lawden spiral.[28] It is an intermediate thrust arc occurring in problems of optimal transfers between Keplerian orbits in a central field. The Lawden spiral was found to be nonoptimal by Robbins[29] and

Kopp and Moyer.[30] The status of the general intermediate-thrust arcs in a central field was given by Archenti et al.,[31-33] many of which are extremal trajectories. The first general studies of singular arcs were made by Kelley,[34-36] Robbins,[29,37] and Contensou[38] in the case of a scalar singular control. They used special control variations and a transformation approach with the objective of finding a suitable state variable which could be considered as control parameter. This method was extended by Kopp and Moyer,[30] Tait,[39] and Kelley et al.[40]; who obtained the generalized Legendre Clebsch condition, also known as the Kelley-Contensou test:

"A necessary condition of extremality for scalar singular control is that

$$(-1)^q \frac{\partial}{\partial u_s}\left[\frac{d^{2q}}{dt^{2q}}\left\{\frac{\partial H(\vec{p},\vec{x},\vec{u},t)}{\partial u_s}\right\}\right] \leq 0 \tag{44}$$

where q is the order of the singular problem and u_s the scalar singular control."

Note that the sign must be opposite if the optimal Hamiltonian H^* is given by a minimization of the dot product $\vec{p}\cdot\dot{\vec{x}}$ rather than a maximization. Also note that this expression assumes sufficient differentiability of the control Hamiltonian H and is reduced to the usual Legendre condition when $q=0$.

The case of a vector singular control was developed by Robbins[37] and Goh[41] and will be discussed in Sec. VI.D. Numerous second-order tests have been set up, such as the Jacobson-Gabasov condition[42,43] (applied at the final point of a singular arc when it is also the final point of the trajectory of interest). Thus, very general surveys can be made.[16,44-50]

The singular arc can be characterized by

> Along a singular arc the domain D_H of Eq. (39) is almost never reduced to a point as in Eq. (38). (45)

Let us consider a first simple example.

Example 9. A generalized Fuller problem.

Control function and control domain: $d^n x/dt^n = u$; (46a)
$-b(t) \leq u \leq a(t)$; $a(t)$, $b(t)$ are integrable and positive

Question:

maximize $I = \int_{t_0}^{t_f} Ax^2 dt$ between given end conditions (46b)

The Mayer form of the problem is the following:

$$x = x_1 \qquad dx/dt = x_2, \ldots \qquad d^k x/dt^k = x_{k+1}, \ldots$$

$$d^{(n-1)}x/dt^{(n-1)} = x_n \tag{47a}$$

the parameter x_{n+1} is related to I: $dx_{n+1}/dt = Ax_1^2$ (47b)

$$H = p_1 x_2 + p_2 x_3 + \ldots + p_{n-1}\cdot x_n + p_n u + p_{n+1} A x_1^2 \tag{47c}$$

and thus:

$$p_n \geq 0 \Rightarrow H^* = p_1 x_2 + \ldots + p_{n-1} x_n + p_n \cdot a(t) + p_{n+1} A x_1^2 \tag{48a}$$

$$p_n \leq 0 \Rightarrow H^* = p_1 x_2 + \ldots + p_{n-1} x_n - p_n \cdot b(t) + p_{n+1} A x_1^2 \tag{48b}$$

According to Eq. (45), the singular arc occurs when $p_n = 0$ over some interval of time. However:

$$\dot{p}_n = -H^*_{x_n} = -p_{n-1}, \ldots \qquad \dot{p}_k = -H^*_{x_k} = -p_{k-1}, \ldots$$

$$(-1)^n d^n p_n/dt^n = -\dot{p}_1 = H^*_{x_1} = -2Ap_{n+1}x_1 \tag{49}$$

which implies

$$(-1)^n d^{2n} p_n/dt^{2n} = 2Ap_{n+1}\cdot u \tag{50}$$

Along the singular arc, p_n and all of its derivatives are zero, hence x_k and $p_k = 0$ for $k = 1,2,\ldots n$ and thus $p_{n+1} \neq 0$ [because of Eq. (37)]. However $\dot{p}_{n+1} = -H^*_{x_{n+1}} = 0$ and, since we want to maximize $x_{n+1,f}$, the transversality conditions lead to:

$$p_{n+1} \equiv p_{n+1\cdot f} > 0 \tag{51}$$

Hence, according to Eq. (44) the singular arc $x \equiv 0$ is of order n, it is optimal if $A < 0$ (indeed it then corresponds to the strong maximum of I, i.e., zero); finally, it is not optimal if $A > 0$; it then corresponds to the minimum rather than the maximum of I.

C. Singular Arcs of Type 1

In the previous section, we considered a control Hamiltonian which was linearly related to some control parameter u; that presentation is insufficiently general and has several drawbacks when compared to the general definition given in Eq. (45).

Let us consider a maneuverability domain such as the concave one of Fig. 2; the maximum principle leads to two possible velocities $\dot{\vec{x}}^*_A$ and $\dot{\vec{x}}^*_B$ (after closure of the domain) and even to all velocities of the segment $\dot{\vec{x}}^*_A$ to $\dot{\vec{x}}^*_B$ after convexization of the domain. That segment corresponds to the relaxation of the control between $\vec{u}^*_A$ and $\vec{u}^*_B$ and also to the linear relation $H = H_1 + uH_2$ of the previous section. Hence that linear relation can either be given directly in the problem of interest, or it can arise indirectly through the closure and the convexization of the maneuverability domain. Thus Eq. (45) is a very general definition of singular arcs.

Let us consider now the domain D_V, projection of D_H on the space of velocities $\dot{\vec{x}}$ and already defined in Eq. (41). It is a closed and convex set of dimension $k(0 \leq k \leq n-1)$ and according to Contensou,[38] the "subboundary" of D_V is defined as its boundary in its own space of dimension k.

According to Ref. 8, a "singular arc of Type 1," is one that satisfies the following three conditions:

> The domain D_H and its projection D_V have almost always the same number of dimensions [i.e., the choice of $\dot{\vec{x}}$ in Eq. (39) implies that of $\vec{p}$) (52a)
>
> The velocity $\dot{\vec{x}}$ belongs almost never to the "subboundary" of D_V (52b)
>
> Only one independent adjoint function $\vec{p}(t)$ corresponds to the trajectory $\vec{x}(t)$ of interest (52c)

Condition (52c) is sometimes called the "normality condition" (remember that $\vec{p}(t)$ and $\lambda\vec{p}(t)$ are not independent when λ is a positive constant, but they are independent if λ is negative).

For scalar singular controls (i.e., when D_H and D_V are of dimension 1) Eq. (52b) is equivalent to the usual condition of a "two-sided variation." It is an important condition, as shown in the following example.

Example 10. Importance of Eq. (52b).

Control function and control domain: $\dot{x}_1 = x_2^2 + \cos u_1$;
$\dot{x}_2 = u_2 + \sin u_1$; $|u_1| \leq \pi$; $0 \leq u_2 \leq 1$ (53a)

End conditions: $x_{1.0} = x_{2.0} = t_0 = 0$; $x_{2f} = 0$; $t_f = 1$ (53b)

Question: maximize x_{1f} (53c)

The generalized Legendre-Clebsch condition of Eq. (44) is not satisfied for the singular solution $x_1 = t$, $x_2 = 0$; however, this solution is optimal because Eq. (52b) is not satisfied and therefore Eq. (44) is not applicable.

D. The Test of Singular Arcs of Type 1

Let us try to write the Kelley-Contensou test of Eq. (44) and its generalization to vector singular controls independently of the control in order to avoid the difficulties underlined in Sec. VI.C.

In the vicinity of an arc with a scalar singular control, the optimal Hamiltonian H^* can be written as:

$$H^* = \sup(H_A^*, H_B^*) \tag{54}$$

where H_A^* corresponds to the portion of the maneuverability domain near $\vec{\dot{x}}_A^*$ (Fig. 2) and H_B^* to the portion near $\vec{\dot{x}}_B^*$. The Kelley-Contensou test [Eq. (44)] can be applied to $H=H_A^*+u(H_B^*-H_A^*)$ with $0\leq u\leq 1$ if the difference $H_B^*-H_A^*$ is sufficiently differentiable. It is even possible to consider $(H_B^*-H_A^*)/g(t)$ instead of $(H_B^*-H_A^*)$ where the function $g(t)$ is any positive integrable function.

Similarly, let us consider a singular arc of Type 1 with k-dimensional domains D_H and D_V. The test can be applied if in a vicinity of the singular arc $\{\vec{p}(t),\vec{x}(t),t\}$ it is possible to write for almost all t:

$$H_1(\vec{p},\vec{x},t)+g(t)\cdot \sup_{i=\{1\ldots k\}}\{|S_i(\vec{p},\vec{x},t)|\}\leq H^*(\vec{p},\vec{x},t)$$

$$\leq H_1(\vec{p},\vec{x},t)+G(t)\cdot \sup_{i=\{1\ldots k\}}\{|S_i(\vec{p},\vec{x},t)|\} \tag{55}$$

with

> $g(t)$ and $G(t)$ are two positive and integrable functions and $H_1(\vec{p},\vec{x},t)$ is continuously differentiable with respect to $\vec{p}$ and $\vec{x}$ at almost all points $\{\vec{p}(t),\vec{x}(t),t\}$ of the singular arc. The k functions $S_i(\vec{p},\vec{x},t)$ are zero along the singular arc (i.e., at points $\vec{p}(t),\vec{x}(t),t$) and sufficiently differentiable in its vicinity. (56)

Thus, if we consider a point $\{\vec{p}(t),\vec{x}(t),t\}$ of the singular arc and the corresponding domain D_H defined in Eq. (39), the velocities $\vec{\dot{x}}$ and $\vec{\dot{p}}$ belonging to D_H are given by k arbitrary parameters λ_i

$$(\dot{\vec{x}},-\dot{\vec{p}})=\frac{\partial H_1}{\partial(\vec{p},\vec{x})}+\sum_{i=1}^{k}\lambda_i\frac{\partial S_i}{\partial(\vec{p},\vec{x})} \tag{57}$$

The test is related to the successive total derivatives (along neighboring extremals) of the function S given by

$$S=\sum_{i=1}^{k}\mu_i S_i(\vec{p},\vec{x},t) \tag{58}$$

These derivatives must be evaluated with constant μ_i and as long as we do not need Eq. (57) we obtain

$$\frac{d^\alpha S}{dt^\alpha}=\sum_{i=1}^{k}\mu_i S_{i\alpha}(\vec{p},\vec{x},t) \tag{59}$$

When Eq. (57) is applied for the first time, we obtain

$$\frac{d^r S}{dt^r}=\sum_{i=1}^{k}\mu_i S_{ir}(\vec{p},\vec{x},t)+\sum_{i=1}^{k}\sum_{j=1}^{k}\mu_i\lambda_j S_{ijr}(\vec{p},\vec{x},t) \tag{60}$$

The test can work only if, in a vicinity of the singular arc, the S_{ir} and S_{ijr} functions are bounded and furthermore the S_{ijr} functions are continuous in terms of $\vec{p}$ and $\vec{x}$. The following necessary conditions of extremality are then obtained.

If r is odd all S_{ijr} must be almost always zero along the singular arc (note that already in this case $S_{ijr}=-S_{jir}$ and thus when $k=1$, i.e., for scalar singular controls, this condition is always satisfied). If r is even, the quadratic form

$$Q(\vec{y},\vec{y})=(-1)^{r/2}\sum_{i=1}^{k}\sum_{j=1}^{k}y_i y_j S_{ijr}\{\vec{p}(t),\vec{x}(t),t\} \tag{61}$$

must be almost always nonpositive definite. When $k=1$, Eq. (61) implies Eq. (44) with $r=2q$.

Note that the functions $g(t)$ and $G(t)$ have two interests: 1) their existence implies Eq. (52b) and 2) they allow extension of the usual domain of the Kelley-Contensou test as we will see in the following examples.

Example 11. A singular arc with an odd r.

Control functions and control domain: $\dot{x}_1=u_1\cdot f_1(t)$; (62)

$\dot{x}_2=u_2\cdot f_2(t)$; $\dot{x}_3=x_1u_2f_2-x_2u_1f_1=x_1\dot{x}_2-x_2\dot{x}_1$;

$|u_1|\leq 1$; $|u_2|\leq 1$; $f_1(t)$ and $f_2(t)$ are positive and integrable

The usual analysis of a singular arc for which

$$x_1=x_2=x_3=0 \qquad p_1=p_2=0 \qquad p_3=1$$

leads to

$$H=u_1f_1\{p_1-p_3x_2\}+u_2f_2\{p_2+p_3x_1\} \tag{63}$$

where

$$u_1=\text{sign}(p_1-p_3x_2) \qquad u_2=\text{sign}(p_2+p_3x_1)$$

$$\dot{p}_1=-u_2f_2p_3 \qquad \dot{p}_2=+u_1f_1p_3$$

and the singular arc of interest corresponds to a vector singular control with $k=2$.

The "optimal Hamiltonian" is:

$$H^*=f_1|p_1-p_3x_2|+f_2|p_2+p_3x_1| \tag{64}$$

Using Eq. (55), we can choose

$$g(t)=\inf\{f_1(t),f_2(t)\} \qquad G(t)=\sup\{f_1(t),f_2(t)\}$$

$$H_1=0 \qquad S_1=p_1-p_3x_2 \qquad S_2=p_2+p_3x_1$$

$$S=\mu_1(p_1-p_3x_2)+\mu_2(p_2+p_3x_1) \tag{65}$$

thus

$$\dot{x}_1=\lambda_1 \qquad \dot{x}_2=\lambda_2 \qquad \dot{x}_3=\lambda_2x_1-\lambda_1x_2$$

$$\dot{p}_1=-\lambda_2p_3 \qquad \dot{p}_2=\lambda_1p_3 \qquad \dot{p}_3=0 \tag{66}$$

and

$$dS/dt=-2\mu_1\lambda_2p_3+2\mu_2\lambda_1p_3 \tag{67}$$

i.e., $S_{1.2.1}=-2p_3=-S_{2.1.1}\neq 0$ and the singular arc of interest is not locally extremal.

Example 12. A singular arc with $r=3$.

Maneuverability domain: $|\dot{x}_1|\leq 1$; $|\dot{x}_2|\leq 1$; $\dot{x}_3=x_1$; $\dot{x}_4=x_2$

$$x_1x_4-x_2x_3-1\leq\dot{x}_5\leq x_1x_4-x_2x_3 \tag{68}$$

The singular arc $\vec{x}=0$; $p_1=p_2=p_3=p_4=0$; $p_5=+1$ corresponds to $k=2$ and is not extremal ($S=\mu_1p_1+\mu_2p_2$, and $S_{1.2.3}=-2p_5\neq 0$, with $\lambda_1=\dot{x}_1$; $\lambda_2=\dot{x}_2$).

Example 13. An extremal singular arc with odd r.

Maneuverability domain: $|\dot{x}_1| \leq 1$; $|\dot{x}_2| \leq 1$

$$x_1^2\dot{x}_2 - x_1x_2\dot{x}_1 - 2x_1^2 - 1 \leq \dot{x}_3 \leq x_1^2\dot{x}_2 - x_1x_2\dot{x}_1 - 2x_1^2 \qquad (69)$$

The singular arc $\vec{x}=0$; $p_1=p_2=0$; $p_3=1$ corresponds to $k=2$ and is extremal in spite of an odd r.

$$S=\mu_1(p_1-x_1x_2p_3)+\mu_2(p_2+x_1^2p_3) \qquad \lambda_1=\dot{x}_1 \qquad \lambda_2=\dot{x}_2$$

hence

$$\mathrm{d}S/\mathrm{d}t=3x_1p_3(\mu_2\lambda_1-\mu_1\lambda_2)+4\mu_1x_1p_3$$

The number r is odd but $S_{1.2.1}$ and $S_{2.1.1}$ are identically zero along the singular arc and the necessary conditions of extremality are satisfied. The consideration of the function $f=2x_3+x_1^2x_2$ which is constant along the singular arc and nonincreasing everywhere else shows that the singular arc of interest is indeed an extremal trajectory.

Example 14. A singular arc with a vector singular control and with r even.

Maneuverability domain: $|\dot{x}_1| \leq 1$; $|\dot{x}_2| \leq 1$

$$Ax_1^2+2Bx_1x_2+Cx_2^2-1 \leq \dot{x}_3 \leq Ax_1^2+2Bx_1x_2+Cx_2^2 \qquad (70)$$

Singular arc of interest: $\vec{x}=0$; $p_1=p_2=0$; $p_3=1$; $t_0 \leq t \leq t_f$

The singular arc corresponds to $k=2$ and leads to $r=2$ and

$$S=\mu_1p_1+\mu_2p_2 \qquad Q=2p_3(Ay_1^2+2By_1y_2+Cy_2^2) \qquad (71)$$

If Q is positive definite or non-negative definite or indefinite, the test works and the singular arc is not extremal; if Q is negative definite, the singular arc can be extremal. We can verify that it is indeed extremal since then x_3 is constant along the singular arc and nonincreasing anywhere else.

It is possible to build many other simple examples showing the importance of each of the remaining conditions of the test (boundedness of the S_{ir} functions, continuity of the S_{ijr} functions, etc.). On the other hand, it is certainly possible to extend the test to the arcs that do not satisfy these last conditions or even Eq. (52b).

E. Singular Arcs of Type 2

In the classification of Ref. 8, the expression "singular trajectories of Type 2" is rather unfortunate: it is applied to trajectories which have several independent adjoint functions $\vec{p}(t)$ and which also satisfy Eq. (52a). These trajectories are very similar to the previous ones or even to the "ordinary trajectories of Pontryagin" given by the usual equations (38). For instance, the singular arc of example 11 is of Type 2 since it also corresponds to the adjoint vector $p_1=p_2=0$; $p_3=-1$. That property is sometimes called abnormality.

However, recall that if in a given bicanonical problem, we consider a candidate solution $\vec{x}(t)$, that solution cannot be ruled out by the Pontryagin maximum principle and its generalization if at least one adjoint function satisfies the transversality conditions related to Eqs. (42) and (43) and, if necessary, the test of singular extremals.

F. Singular Arcs of Type 3

These trajectories no longer satisfy Eq. (52a), as shown in the following example:

Example 15. A singular trajectory of Type 3.

Control functions: $\dot{x}_1=u_1(1+u_3x_2)$; $\dot{x}_2=u_2(1+u_3x_2)$ (72a)

Control domain: $|u_1|+|u_2| \leq 1$; $|u_3| \leq 1$ (72b)

End conditions: $x_{1.0}=x_{2.0}=t_0=0$; x_{1f} is free; $x_{2f}=0$; t_f is given (72c)

Question: maximize $I=x_{1f}$ (72d)

Hence

$$H=(1+u_3x_2)(p_1u_1+p_2u_2) \qquad (73)$$

$$H^*=(1+|x_2|)\cdot\sup\{|p_1|,|p_2|\} \qquad (74)$$

with

$$x_2 \neq 0 \Rightarrow u_3=\mathrm{sign}x_2 \qquad (75)$$

$$|p_i|>|p_j| \Rightarrow u_i=\mathrm{sign}p_i \qquad u_j=0 \qquad (76)$$

and

$$\dot{p}_1=-\partial H^*/\partial x_1=0 \qquad (77)$$

$$\dot{p}_2=-\partial H^*/\partial x_2=-(\mathrm{sign}x_2)\cdot\sup\{|p_1|;|p_2|\} \qquad (78)$$

When $x_2=0$, Eq. (39) leads to:

$$|\dot{p}_2| \leq \sup\{|p_1|;|p_2|\} \qquad (79)$$

Let us consider the candidate solution $x_1=t$; $x_2=0$, which corresponds to

$$\dot{x}_1=u_1=1 \qquad \dot{x}_2=u_2=0 \qquad (80)$$

and, thus, to

$$p_1>0 \qquad p_1 \geq |p_2| \qquad (81)$$

If $p_1>|p_2|$, the corresponding domain D_H defined in Eq. (39) is given by

$$\dot{x}_1=1 \qquad \dot{x}_2=0 \qquad \dot{p}_1=0 \qquad |\dot{p}_2| \leq p_1 \qquad (82)$$

$\dot{\vec{p}}$ is not defined by the choice of $\dot{\vec{x}}$ and thus Eq. (52a) is not satisfied: D_H has a dimensionality of 1 and its projection D_V a dimensionality of 0. On the other hand, it is possible to choose, for instance, $p_2 \equiv 0$ and the usual Pontryagin conditions cannot rule out the candidate solution.

However, it is possible to improve the analysis[8] if we consider an "auxiliary Hamiltonian" $L(\vec{p},\vec{x},t)$ which is a measurable function of $\vec{p}$, $\vec{x}$, and t, locally Lipschitz with respect to $\vec{p}$ and $\vec{x}$, and verifying for almost all t in some vicinity of the trajectory $\vec{x}(t)$ of interest:

$$L(\vec{p},\vec{x},t) \leq H^*(\vec{p},\vec{x},t) \qquad (83)$$

$$L(\vec{p},\vec{x}(t),t) \geq \vec{p}\cdot\dot{\vec{x}}(t) \qquad (84)$$

Equation (83) implies that L corresponds to a maneuverability smaller than the initial one; however, Eq. (84) implies that this smaller maneuverability agrees with the trajectory of interest. Thus, we can guess that to any L function there must correspond an adjoint function $\vec{p}(t)$ satisfying conditions corresponding to the generalized Pontryagin conditions of Eqs. (37-39). For instance, we may choose

$$L=(1+x_2)\cdot\sup\{|p_1|,|p_2|\} \qquad (85)$$

which implies

$$\dot{p}_1=0 \qquad \dot{p}_2=-\partial L/\partial x_2=-\sup\{|p_1|,|p_2|\} \qquad (86)$$

and Eq. (81) can be satisfied only if $t_f \leq 2$. Thus, the singular arc of interest is nonextremal as soon as $t_f>2$.

We can verify that when $t_f > 2$, there are two symmetrical optimal solutions, one of them being the following:

$$0 \le t \le (t_f - 2)/2: \qquad \begin{cases} u_1 = 0; u_2 = u_3 = 1 \\ x_1 = 0; x_2 = -1 + \exp\{t\} \end{cases}$$

$$(t_f - 2)/2 < t < (t_f + 2)/2:$$

$$\begin{cases} u_1 = 1; u_2 = 0; u_3 = 1 \\ x_1 = [(2t - t_f + 2) \cdot \exp\{(t_f - 2)/2\}]/2 \\ x_2 = -1 + \exp\{(t_f - 2)/2\} \end{cases} \tag{87}$$

$$(t_f + 2)/2 \le t \le t_f: \qquad \begin{cases} u_1 = 0; u_2 = -1; u_3 = 1 \\ x_1 = 2 \cdot \exp\{(t_f - 2)/2\} \\ x_2 = -1 + \exp\{t_f - t\} \end{cases}$$

The final value of x_1, i.e., $2 \cdot \exp\{(t_f - 2)/2\}$, is indeed larger than the value t_f given by the singular arc.

G. Chattering Arcs of the Second Kind

In Note 6 of Sec. IV, we discussed chattering arcs of the first kind: the optimal control chatters at an arbitrary rapid rate between two or several values because the maneuverability domain is not convex. This is a rather artificial singularity. On the contrary, chattering arcs of the second kind are concrete; they arise before the beginning and after the end of singular arcs of order two or more.[7,71-77] Their prototype is given by the Fuller problem.[7,72-74]

Example 16. The Fuller problem (a chattering arc of the second kind).

$$\text{minimize} \int_{t_0}^{t_f} x^2 \mathrm{d}t \text{ between given end conditions} \tag{88}$$

the parameter $\ddot{x}$ being limited by $|\ddot{x}| \le 1$. The Mayer form of the problem is

$$x_1 = x \qquad x_2 = \dot{x} \qquad x_3 = \int_{t_0}^{t} x^2(\theta) \cdot \mathrm{d}\theta \tag{89a}$$

$$\dot{x}_1 = x_2 \qquad -1 \le \dot{x}_2 \le +1 \qquad \dot{x}_3 = x_1^2 \tag{89b}$$

$$x_{1.0}, x_{2.0}, x_{3.0}, \text{ and } t_0 \text{ are given as well as } x_{1f}, x_{2f}, t_f \tag{89c}$$

$$x_{3f} \text{ is free and to be minimized } (I = -x_{3_f}) \tag{89d}$$

$$H = p_1 x_2 + p_2 \dot{x}_2 + p_3 x_1^2 \tag{90}$$

$$H^* = p_1 x_2 + |p_2| + p_3 x_1^2 \tag{91}$$

with

$$\dot{x}_2 = \mathrm{sign} p_2 \quad \text{if} \quad p_2 \ne 0$$

$$\dot{x}_2 \text{ arbitrary in } [-1, +1] \quad \text{if} \quad p_2 = 0 \tag{92}$$

and

$$\dot{p}_2 = -\partial H^*/\partial x_2 = -p_1 \quad \ddot{p}_2 = -\dot{p}_1 = \partial H^*/\partial x_1 = 2p_3 x_1 \tag{93a}$$

$$\dot{p}_3 = -\partial H^*/\partial x_3 = 0 \qquad \mathrm{d}^3 p_2/\mathrm{d}t^3 = 2p_3 \dot{x}_1 = 2p_3 x_2 \tag{93b}$$

$$\mathrm{d}^4 p_2/\mathrm{d}t^4 = 2p_3 \dot{x}_2 = 2p_3 \cdot \mathrm{sign} p_2 \quad (\text{if} \quad p_2 \ne 0) \tag{93c}$$

Since $I = -x_{3f}$, we have $p_{3f} \le 0$. The case $p_{3f} = 0$ and then $p_3 \equiv 0$ has no singularity but concerns only infinitely rare end conditions. The general case is $p_3 < 0$ and we will choose $p_3 = -0.5$ since the adjoint functions can always be multiplied by a positive and constant factor.

Equation (93c) becomes

$$\mathrm{d}^4 p_2/\mathrm{d}t^4 = -\ddot{x} = \begin{cases} -\mathrm{sign} p_2 & \text{if} \quad p_2 \ne 0 \\ \text{any value in } [-1; +1] & \text{if} \quad p_2 = 0 \end{cases} \tag{94}$$

which has three types of solutions:

$$p_2 \ge 0: \qquad \mathrm{d}^4 p_2/\mathrm{d}t^4 = -1 \qquad \ddot{x} = 1 \tag{95}$$

$$p_2 \le 0: \qquad \mathrm{d}^4 p_2/\mathrm{d}t^4 = 1 \qquad \ddot{x} = -1 \tag{96}$$

$$\text{the singular solution } p_2 \equiv 0: \qquad x \equiv \ddot{x} \equiv 0 \tag{97}$$

p_2 and its three first derivatives $(-p_1, -x_1, -x_2)$ are continuous so that it is easy to match solutions of Eqs. (95) and (96) with a switch at $p_2 = 0$; however, it is impossible to match directly the singular solution of Eq. (97) and an ordinary arc. That function can be made indirectly through an infinite number of switches at $p_2 = 0$ (Fig. 3).

The switching instants t_n are in a geometrical progression of ratio $J = 4.13016$ after the final instant t_s of the singular arc (or before its initial instant). Let us put

$$t - t_s = \theta \qquad t_n - t_s = \theta_n \tag{98}$$

hence

$$\theta_{n+1} = J\theta_n \qquad \theta_n = J^n \theta_0 \tag{99}$$

and from Fig. 3

$$\theta \in (\theta_n, \theta_{n+1}) \text{ implies } \ddot{x} = (-1)^n = -\mathrm{d}^4 p_2/\mathrm{d}t^4 \tag{100}$$

The successive integrations of $\ddot{x}$ give x_2, x_1, p_1, and $-p_2$ and thus

$$x_2(J\theta) = -Jx_2(\theta) \tag{101}$$

$$x_1(J\theta) = -J^2 x_1(\theta) \tag{102}$$

$$p_1(J\theta) = -J^3 p_1(\theta) \tag{103}$$

$$p_2(J\theta) = -J^4 p_2(\theta) \tag{104}$$

The value of J is given by the condition $p_2 = 0$ at all switches, i.e., when $\theta = \theta_n$; it leads to

$$J^4 - 3J^3 + 4J^2 - 3J + 1 = 0 \tag{105}$$

i.e.,

$$J = \{3 + \sqrt{33} + (26 + 6\sqrt{33})^{1/2}\}/4 = 4.13016 \tag{106}$$

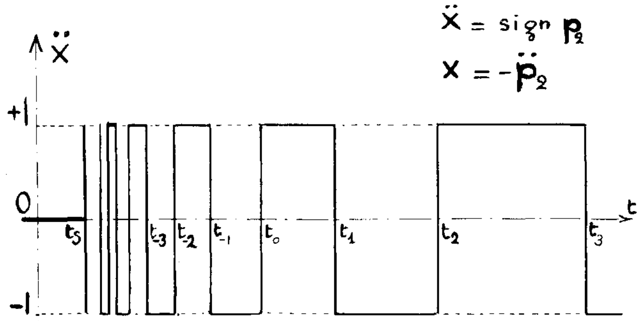

Fig. 3 Chattering arc of the second kind. $\ddot{x}$ has an infinite number of switches at the instants t_n with: $t_n - t_s = J^n (t_0 - t_s)$ where $J = 4.13016$ and $J^4 - 3J^3 - 4J^2 - 3J + 1 = 0$. ($t_0$ is at a switch and is not the initial time; t_s is the terminal time of the singular arc and also a point of accumulation of switches.)

Equations (101-104) imply that x_2, x_1, p_1, and p_2 goes to zero when $\theta \to 0$, i.e., when $t \to t_s$. Thus these four parameters are continuous and it is easy to verify that all other Pontryagin requirements are satisfied.

Thus, the singular solution and its chattering escape constitute a true Pontryagin solution and we can note that it is also an extremal as are all other Pontryagin solutions of example 16. Indeed $\vec{x}(t), \vec{p}(t)$ is such a solution in the Mayer form: the function $F = \vec{p}(t) \cdot \{\vec{x} - \vec{x}(t)\}$ is constant along the solution of interest and nonincreasing anywhere else.

Let us consider, for instance, the following end conditions:

$$x_{1.0} = 2;\ x_{2.0} = -2;\ x_{3.0} = t_0 = 0;\ x_{1f} = x_{2f} = 2;\ t_f = 8 \qquad (107)$$

Then, the optimal solution is symmetrical with respect to $t = 4$ and leads to the optimal $x_{3f} = 3.03046$, which has a singular arc between t_s and $8 - t_s$ with $t_s = 3.43039$ and its first switch occurs at $t = 0.057765$. The instants t_s and $8 - t_s$ are points of accumulation of switches and the theoretical number of switches is infinite.

We can note that: 1) Piecewise continuous controls cannot give the optimal solution. 2) The optimal solution is physically impossible. Fortunately, excellent piecewise continuous solutions with a small number of switches are available and convergence is extremely rapid.[20] 3) These chattering arcs of the second kind appear for very general end conditions and give a large generality to the singular arcs.

Let us now consider an unsymmetrical chattering arc of the second kind.

Example 17. Unsymmetrical chattering arc of the second kind.

$$\text{minimize} \int_{t_0}^{t_f} x^2 \cdot dt \ \text{ between given end conditions} \qquad (108)$$

d^2x/dt^2 being bounded by $-b \le \ddot{x} \le a$ with $a > 0$, $b > 0$. Equation (94) becomes

$$d^4p_2/dt^4 = -\ddot{x} = \begin{cases} -a & \text{if } p_2 > 0 \\ \text{any value in } [-a, +b], & \text{if } p_2 = 0 \\ +b & \text{if } p_2 < 0 \end{cases} \qquad (109)$$

which has three types of solutions:

$$p_2 \ge 0: \qquad d^4p_2/dt^4 = -a \qquad \ddot{x} = +a \qquad (110)$$

$$p_2 \le 0: \qquad d^4p_2/dt^4 = +b \qquad \ddot{x} = -b \qquad (111)$$

$$\text{the singular solution:} \qquad p_2 \equiv 0 \qquad x \equiv \ddot{x} \equiv 0 \qquad (112)$$

It leads to chattering arcs with two alternate ratios

$$\text{between } t_{2n} \text{ and } t_{2n+1}\text{: } \ddot{x} = a \text{ and } (t_{2n+1} - t_s)/(t_{2n} - t_s) = \alpha$$

$$\text{between } t_{2n-1} \text{ and } t_{2n}\text{: } \ddot{x} = -b \text{ and } (t_{2n} - t_s)/(t_{2n-1} - t_s) = \beta \qquad (113)$$

α and β are only functions of the ratio b/a (Fig. 4). They are given by the following four analytical relations:[20]

$$\sqrt{(\alpha\beta)} = s \ge 4.13016 \qquad \alpha + \beta = \frac{s^4 - s^3 - 2s^2 - s + 1}{s^2 + s + 1}$$

$$\frac{b}{a} = \frac{\alpha - R}{\beta - R}\sqrt{\frac{\beta}{\alpha}} \qquad R = \frac{s^4 + 3s^3 + 4s^2 + s + 1}{s^4 + s^3 + 4s^2 + 3s + 1} \qquad (114)$$

These unsymmetrical chattering arcs are useful in understanding the usual chattering arcs that surround the optimal singular arcs of the second order (i.e., the intermediate thrust arcs of space dynamic[31-33]). Indeed, in a small neighborhood of the junction, the succession of switches is infinitely close to the alternative geometrical progression of Eq. (113).

Example 18. Chattering arcs of order n.

$$\text{minimize} \int_{t_0}^{t_f} x^2 dt \ \text{ between given end conditions} \qquad (115)$$

the n derivative d^nx/dt^n being bounded by $|d^nx/dt^n| \le 1$. The Mayer form of the problem is a simple extension of Eq. (89) and Eq. (94) becomes:

$$(-1)^{n+1} d^{2n}p_n/dt^{2n} = \text{sign}p_n = d^nx/dt^n \qquad (116)$$

The singular arc corresponds to $p_n \equiv 0$ and, if n is odd, it is possible to escape without chattering, with $p_n = \pm (t - t_s)^{2n}/(2n)!$ after t_s. However, when $n > 1$, that one parameter possibility is not sufficiently general and can only lead to particular final conditions.

If $n \ge 2$, it is also possible to escape from the singular arc by a chattering arc of the second kind similar to that of Fig. 3 with a geometrical progression of ratio J_n satisfying:

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \Big/ (1 + J_n^k) = 0 \qquad (J_n > 1) \qquad (117)$$

Thus, $J_2 = 4.13016$, $J_3 = 1.7369$, $J_4 =$ either 1.3509 or 9.9812, etc. Equation (116) on the sign of p_n rules out some of the roots of Eq. (117) and, for a given n, the number of remaining roots is either $n/2$ if n is even or $(n-1)/2$ if n is odd.

When $n \ge 3$, these chattering arcs with a geometrical progression are not sufficiently general (each has only two parameters) and the general solution uses a mixing of the different possibilities mentioned above,[20] two parameters being given by each chattering arc and, for n odd, one

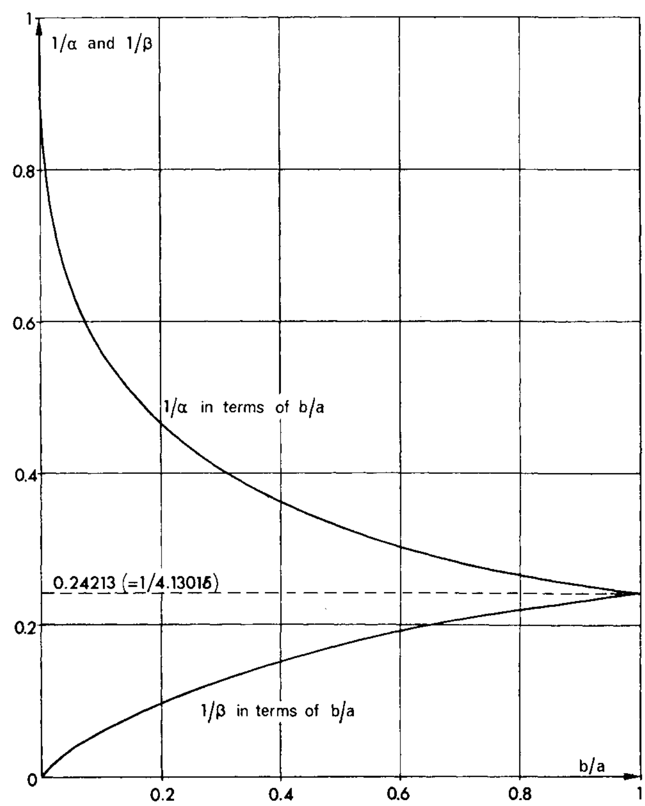

Fig. 4 $1/\alpha$ and $1/\beta$ in terms of ratio b/a for unsymmetrical chattering arcs of the second kind.

parameter being given by the direct escape. This consequently provides the n parameters necessary to reach arbitrary final conditions and, hence, between given terminal conditions on $t,x,\dot{x},\ldots,d^{(n-1)}x/dt^{(n-1)}$, the optimal solution uses a singular arc along $x=0$ when $t_f - t_0$ is sufficiently large.

Note that all of these solutions have the same $\vec{x}$ and the same $\vec{p}$ along the singular arc, and hence that singularity destroys the deterministic character of the usual Pontryagin equations (38). The same initial $\vec{x}$ and $\vec{p}$ can lead to various final conditions.

In case of a vector singular control, the chattering arcs of the second kind become even more complex and the general theory of this phenomenon is very difficult.[20]

VII. Conclusions

Singularities in optimization problems is a very broad subject even if restricted to the analysis of maneuverable and deterministic systems governed by ordinary differential equations. The substitution of the notion of maneuverability for that of control leads to a simple and unified presentation.

In addition to usual topological singularities and to an artificial singularity disappearing in the Mayer form of the problem, there exist singularities related to the optimization: singular arcs and chattering arcs of the second kind. Together, these arcs play a central role in many optimization problems: they appear as part of general solutions and not only as particular solutions (as are the parabolas among Keplerian orbits).

It would be interesting to enlarge the analysis to numerical questions, to nonbicanonical problems (with, for instance, a forbidden zone or with discontinuities of the maneuverability with respect to the state), and to singularities of second order tests: the "adjoint matrix" of these tests have negative jumps at each switch and at the beginning of singular arcs,[48] and there may be an infinite number of switches.

Under these circumstances, it is not surprising that the optimization of systems is extremely complex and difficult to understand, even when they are of a deterministic nature and without distributed parameters.

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